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DYNAMICS OF A BEAM MOVING OVER MULTIPLE SUPPORTS

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Abstract—The equations of motion of a beam moving over multiple supports are formulated based on Hamilton's principle and the assumed mode method. The supports are either roller supports which impart motion to the beam or frictionless supports with the beam being pushed or pulled over them. In both cases, the supports are regarded as very stiff springs acting on the moving beam. A feature of the present formulation is that its complexity does not increase with an increased number of supports. Results of numerical simulations are presented for various prescribed motions of the beam.

1. INTRODUCTION

Beams acted upon by moving loads have received a good deal of attention for a long time in connection with the machining process and applications in the behavior of railway tracks and bridges under moving loads. The classical solution of a beam subjected to a constant moving load was presented by Timoshenko (1992). Subsequent studies by Nelson and Conover (1971), Benedetti (1974), Steele (1967), Florence (1965), and Katz *et al.* (1987) include the effects of elastic foundation, moving mass and deflection dependent moving loads. A related problem involving beams of infinite length moving over supports, or acted upon by moving loads was presented in a series of papers by Adams (1976, 1978a, b, 1979), Adams and Bogy (1975) and Adams and Manor (1981). The present problem for the dynamics of a beam of finite length moving over supports was first presented by Buffinton and Kane (1985) using the method of Kane's dynamics. As pointed out in their work, the forces exerted on the beam by the supports have magnitudes depending on the motion of the beam. Moreover, the beam moves not only relative to an inertial frame but also relative to its support. The external forces that cause the motion of the beam, however, were not stated explicitly in the paper.

In the present study, equations of motion in matrix form for a finite beam moving over multiple supports are formulated using Hamilton's principle and the assumed mode method. The external forces that cause the motion of the beam are either in the form of the frictional forces supplied by a roller which acted as a support at the same time, or external forces applied at one end of the beam. Supports which are not in the form of a roller are assumed to be frictionless. These supports are regarded as very stiff springs acting on the moving beam. It will be shown in the formulation that the complexity of the formulation does not increase with the increased number of supports.

2. THEORY AND FORMULATIONS

The beam considered is assumed to be a uniform beam of length L moving horizontally over two supports. Two different combinations of supports will be considered for the

Fig. 1. A beam moving over two frictionless supports.

present study. The first combination, shown in Fig. 1, consists of two frictionless supports. Horizontal forces are applied at the left end of the beam to pull or push the beam over these supports. The second combination, shown in Fig. 2, consists of a roller support and a frictionless support. The beam is rolled forward or backward over the supports by the frictional forces developed between the roller and the beam. The assumptions made in the development of the following formulation are that transverse deflections are small so that the dynamic behavior of the beam is governed by the Euler beam theory. Moreover, all the transverse deflections and the axial motion of the beam occur in the same plane defined by $\mathbf{n}_1$ and $\mathbf{n}_2$ unit vectors fixed in the inertial frame. A set of mutually perpendicular unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ is assumed to be fixed in the beam with the $\mathbf{i}$ vector parallel to the $\mathbf{n}_1$ vector and the origin located at the left end of the beam. Flexibility of the beam in the axial direction $\mathbf{i}$ is assumed to be negligible compared to the lateral direction $\mathbf{j}$.

The position vector of a general point P on the deformed beam is given by

$$\mathbf{p} = x\mathbf{i} + w\mathbf{j}. \tag{1}$$

The velocity at the point is

$$\mathbf{v}_p = U\mathbf{i} + \dot{w}\mathbf{j}, \tag{2}$$

where $\dot{w} = \mathrm{d}w/\mathrm{d}t$ and $U\mathbf{i}$ is the prescribed velocity of the beam at $x = 0$ as a result of the external applied forces or the frictional forces developed between the roller and the beam. Due to the assumption of axial rigidity, U is also the $\mathbf{i}$ component of the velocity for every point along the beam.

The kinetic energy T of the beam is

$$T = \tfrac{1}{2}m \int_0^L (U^2 + \dot{w}^2)\,\mathrm{d}x, \tag{3}$$

where m is the mass of the beam per unit length.

Assuming Euler's beam theory, the elastic strain energy of the beam due to bending is

$$V_\varepsilon = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \mathrm{d}x, \tag{4}$$

where E and I are the Young's modulus and the central principle second moment of area of the cross-section of the beam about the $\mathbf{k}$ axis, respectively.

The supports are regarded to be very stiff springs of stiffness k. The potential energy due to the supports is

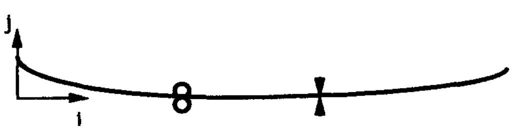

Fig. 2. A beam moving over a roller support and a frictionless support.

$$V_s = \tfrac{1}{2}k\{w^2(a)+w^2(b)\}, \tag{5}$$

where $x = a$ and $x = b$ are the coordinates of the supports relative to the moving beam. These relative locations of the supports change with the motion of the beam. $w(a)$ and $w(b)$ are the deflections of the beam at $x = a$ and $x = b$, respectively. It can be seen from the above expression that increasing the number of supports will only increase the number of terms in eqn (5). It will not directly affect the expressions for kinetic energy, strain energy, and the following expression for the potential energy due to the axial forces of the beam.

The potential energy due to the axial forces, f_x, is

$$V_a = \frac{1}{2}\int_0^L f_x\left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x. \tag{6}$$

The axial forces, f_x, will be dependent on the external forces applied to the beam. For the case of a beam moving over a pair of frictionless supports shown in Fig. 1, the right end of the beam is stress free as the beam is being pulled or pushed over the supports by external forces applied at the left end of the beam. The axial forces for the remaining part of the beam are

$$\begin{aligned} f_x &= -m\int_x^L \dot{U}\,\mathrm{d}r \\ &= -m\dot{U}(L-x). \end{aligned} \tag{7}$$

For a beam being rolled over the supports shown in Fig. 2, both the left and right ends of the beam are stress free as there are no external applied forces at these two ends. With the roller support at $x = a$, the axial forces within the beam are

$$\begin{aligned} f_x &= -m\int_x^L \dot{U}\,\mathrm{d}r \\ &= -m\dot{U}(L-x) \end{aligned} \tag{8}$$

for $a \leqslant x \leqslant L$ and

$$\begin{aligned} f_x &= m\int_0^x \dot{U}\,\mathrm{d}r \\ &= m\dot{U}x \end{aligned} \tag{9}$$

for $0 \leqslant x < a$.

The discontinuity of f_x at $x = a$ is due to the presence of the frictional forces at $x = a$ developed between the roller and the beam. The difference in f_x at $x = a$ is the frictional force required to carry out the prescribed motion of the beam.

The quantity w is expressed as

$$w = \sum_{i=1}^{n} q_i(t)\phi_i(x), \tag{10}$$

where ϕ_i are spatial functions that satisfy the geometric boundary conditions at the two ends of the beam. For the present study, ϕ_i are assumed to be the normalized modal functions for the vibration of a uniform unrestrained beam. The assumed functions are

$$\phi_1(x) = 1, \tag{11}$$

$$\phi_2(x) = \sqrt{3}\left(1 - \frac{2x}{L}\right), \tag{12}$$

$$\phi_i(x) = \cos\frac{\lambda_{i-2}x}{L} + \cosh\frac{\lambda_{i-2}x}{L} - \gamma_{i-2}\left[\sin\frac{\lambda_{i-2}x}{L} + \sinh\frac{\lambda_{i-2}x}{L}\right] \quad (i = 3, \ldots, n), \tag{13}$$

where

$$\gamma_j = \frac{\cos\lambda_j - \cosh\lambda_j}{\sin\lambda_j - \sinh\lambda_j} \tag{14}$$

and $\lambda_1, \ldots, \lambda_n$ are the consecutive roots of the transcendental equation

$$1 - \cos\lambda\cosh\lambda = 0. \tag{15}$$

ϕ_1 and ϕ_2 correspond to the rigid body translation and rotation of an unrestrained beam. The assumed form of w enables the kinetic energy, the strain energy and the potential energy to be expressed in matrix form. For the case in Fig. 1 with the external forces applied at the left end of the beam, the matrix forms of the energy involving w are

$$T = \tfrac{1}{2}m\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{M}\dot{\mathbf{q}}, \tag{16}$$

$$V_\varepsilon = \tfrac{1}{2}EI\mathbf{q}^{\mathrm{T}}\mathbf{H}\mathbf{q}, \tag{17}$$

$$V_s = \tfrac{1}{2}k\mathbf{q}^{\mathrm{T}}(\mathbf{\Phi}_a + \mathbf{\Phi}_b)\mathbf{q} \tag{18}$$

$$= \tfrac{1}{2}k\mathbf{q}^{\mathrm{T}}\mathbf{\Phi}\mathbf{q},$$

$$V_a = \tfrac{1}{2}m\dot{U}\mathbf{q}^{\mathrm{T}}\mathbf{Y}\mathbf{q} - \tfrac{1}{2}m\dot{U}L\mathbf{q}^{\mathrm{T}}\mathbf{Q}\mathbf{q}, \tag{19}$$

where **M**, **H**, **Y**, **Q** and $\mathbf{\Phi}_m$ are matrices defined as

$$(\mathbf{M})_{ij} = \int_0^L \phi_i\phi_j\,\mathrm{d}x, \tag{20}$$

$$(\mathbf{H})_{ij} = \int_0^L \phi_i''\phi_j''\,\mathrm{d}x, \tag{21}$$

$$(\mathbf{Y})_{ij} = \int_0^L x\phi_i'\phi_j'\,\mathrm{d}x, \tag{22}$$

$$(\mathbf{Q})_{ij} = \int_0^L \phi_i'\phi_j'\,\mathrm{d}x, \tag{23}$$

$$(\mathbf{\Phi}_m)_{ij} = \phi_i(x = m)\phi_j(x = m). \tag{24}$$

It can be seen from the above expressions that **M**, **H**, **Y**, **Q** and $\mathbf{\Phi}_m$ are symmetric matrices. All the matrices except $\mathbf{\Phi}_m$ are independent of time. $\mathbf{\Phi}_m$ need to be updated as the beam is moving over the supports. ϕ_i' and ϕ_i'' denote the first and second derivatives of ϕ_i with respect to x. $\mathbf{q}$ and $\dot{\mathbf{q}}$ are $n \times 1$ column vectors consisting of q_i and $\dot{q}_i$ respectively.

The Lagrangian of the beam involving w can be expressed as

$$L = T - V_\varepsilon - V_a - V_s. \tag{25}$$

The Euler–Lagrange equation for a beam moving over supports shown in Fig. 1 is

$$m\mathbf{M}\ddot{\mathbf{q}} + (EI\mathbf{H} + m\dot{U}\mathbf{Y} + k\mathbf{\Phi} - m\dot{U}L\mathbf{Q})\mathbf{q} = 0. \tag{26}$$

For the case of a beam being rolled over supports shown in Fig. 2, the matrix expressions for T, V_ε and V_s remain the same. The matrix expression for V_a is

$$V_a = \tfrac{1}{2}m\dot{U}\mathbf{q}^T\mathbf{Y}\mathbf{q} - \tfrac{1}{2}m\dot{U}L\mathbf{q}^T\mathbf{Q}_1\mathbf{q}, \tag{27}$$

where $\mathbf{Q}_1$ is a symmetric matrix defined as

$$(\mathbf{Q}_1)_{ij} = \int_a^L \phi_i'\phi_j'\,\mathrm{d}x. \tag{28}$$

The corresponding Euler–Lagrange equation for a beam moving over supports shown in Fig. 2 is

$$m\mathbf{M}\ddot{\mathbf{q}} + (EI\mathbf{H} + m\dot{U}\mathbf{Y} + k\mathbf{\Phi} - m\dot{U}L\mathbf{Q}_1)\mathbf{q} = 0. \tag{29}$$

3. RESULTS AND SIMULATIONS

Sinusoidal longitudinal motions

The equations of motion generated in the preceding sections can be included in numerical simulation programs for investigating the response of a beam undergoing various prescribed motions. The numerical integrations are performed using the fourth order Runge–Kutta method. For the present numerical simulations, $L = 1$ m, $m = 1$ kg m^{-1}, and $EI = 1$ N m^{-2}. The beam is assumed to be moving over two supports with a separation of $D = 0.25$ m. The initial configuration of the beam is such that the two supports are located symmetrically at $a = 0.375$ m and $b = 0.626$ m. The initial shape of the beam is described by

$$w = \phi_1 q_1 + \phi_3 q_3. \tag{30}$$

q_1 and q_3 are determined from the conditions that $w = 0$ m for both supports at $a = 0.375$ m and $b = 0.625$ m, and $w = 0.01$ m at $x = 0$ m and at $x = L$ m. Since ϕ_1 corresponds to the rigid body translation, the initial shape of the beam is the first symmetric flexural mode shape of an unrestrained beam passing through the two supports. The natural frequencies for this beam without longitudinal motion are 16.246 and 20.771 Hz for the first and second modes (Buffinton and Kane, 1985). This prescribed initial shape of the beam is different from the prescribed initial shape of the beam in the work by Buffinton and Kane (1985), which is the deformed shape of a beam by a statically applied, uniform load such that the deflection at the left end of the beam is 0.01 m. The present initial shape of the beam is assumed so that the step of optimal curve fitting for the initial deformed shape of the beam can be avoided. Moreover, a minimum of four-term approximation for w is required to enforce general prescribed initial displacements at $x = 0$, $x = L$, and the constraint of zero displacement at $x = a$ and $x = b$. Any optimal curve fitting for the prescribed initial deformed shape of the beam will require that the number of terms in the approximate function for w be four or greater than four.

For a prescribed longitudinal sinusoidal motion of the beam, the variation of $x = a$ is assumed to be

$$a = 0.375 - A \sin \Omega t \text{ m}. \quad (31)$$

For a beam moving over supports shown in Fig. 1, the transverse displacements at $x = 0$ and $x = a$ of the beam for $\Omega = 20$ rad s^{-1} and $A = 0.05$ are shown in Fig. 3 using six-term approximation for w ($n = 6$). It can be seen that the tip displacements at $x = 0$ for $k = 10^5$, $k = 10^6$ and $k = 10^7$ are almost identical. On the other hand, the displacement at the left support ($x = a$) is almost negligible for $k = 10^6$ and $k = 10^7$. The displacement at the right support ($x = b$) is also found to be negligible for these two assumed values of stiffness. The execution time for the numerical integrations is found to increase with increased value of k. For the following simulations, k is chosen to be 10^6 N m^{-1}. Numerical simulations for such a value of k can be performed quite efficiently within a few minutes, depending on the prescribed longitudinal motion, on a 486-33 MHz personal computer.

The tip displacement at the left end of the beam for $\Omega = 10$, 20 and 22 rad s^{-1} and $A = 0.05$ are shown in Figs 4–6. In Fig. 4, the beam shows a stable behavior, the same as the stable behavior of the beam reported by Buffinton and Kane (1985). Moreover, the displacements for $n = 6$ and $n = 7$ are close to each other. Unstable behaviors are observed in both Figs 5 and 6, the same as the behaviors predicted by Buffinton and Kane (1985). In Fig. 7, A is reduced to 0.025 for $\Omega = 20$ rad s^{-1} and the beam is found to exhibit a stable behavior, consistent with the corresponding predicted results of Buffinton and Kane (1985).

A general stability analysis can be performed by the method presented by Buffinton and Kane (1985) using Floquet's theory. Such a laborious analysis will not be performed for the present study. The present study, however, will attempt to show the different behavior of the beam if the longitudinal sinusoidal motion of the beam is imparted by a roller as shown in Fig. 2.

For the beam moving over supports shown in Fig. 2, the motion of the beam is imparted by the roller support. The computations are in general more time consuming as the matrix $\mathbf{Q}_1$ needs to be updated during the prescribed longitudinal motion for this type of support configuration. The tip displacements of the beams for $A = 0.05$, $k = 10^5$ N m^{-1}, $\Omega = 20$, 10 and 22 rad s^{-1} are shown in Figs 8–10. For $\omega = 20$ rad s^{-1}, the beam shows an unstable behavior. However, the rate of increase of the tip displacement is very much slower than the corresponding rate of increase for the tip displacement shown in Fig. 5. The behaviors of the beam for $\Omega = 10$ and 22 rad s^{-1}, shown in Figs 9–10, are dramatically

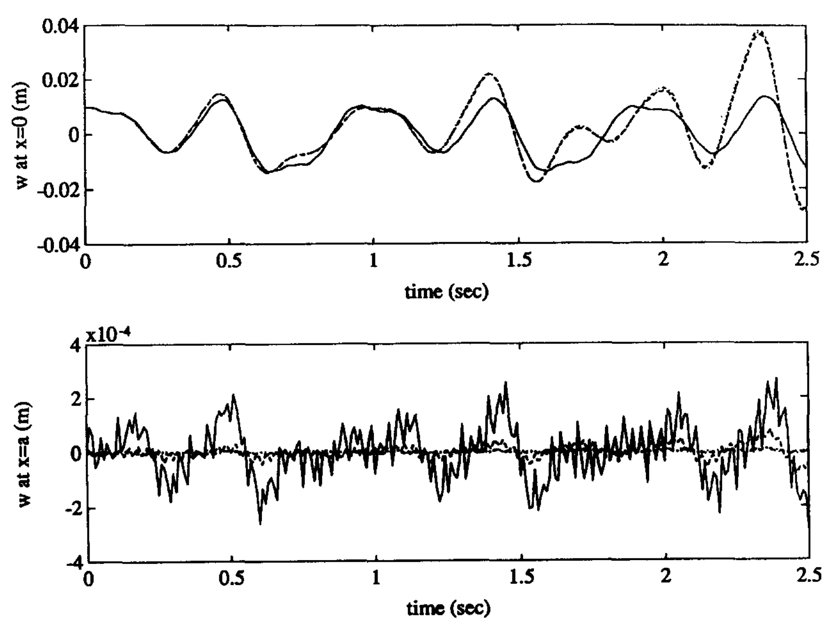

Fig. 3. Tip displacements for a beam moving over supports shown in Fig. 1, $\Omega = 20$ rad s^{-1}, ——, $k = 1 \times 10^4$ N m^{-1}, - - - -, $k = 1 \times 10^5$ N m^{-1}, · · · · · ·, $k = 1 \times 10^6$ N m^{-1}, - · - · - · -, $k = 1 \times 10^7$ N m^{-1}.

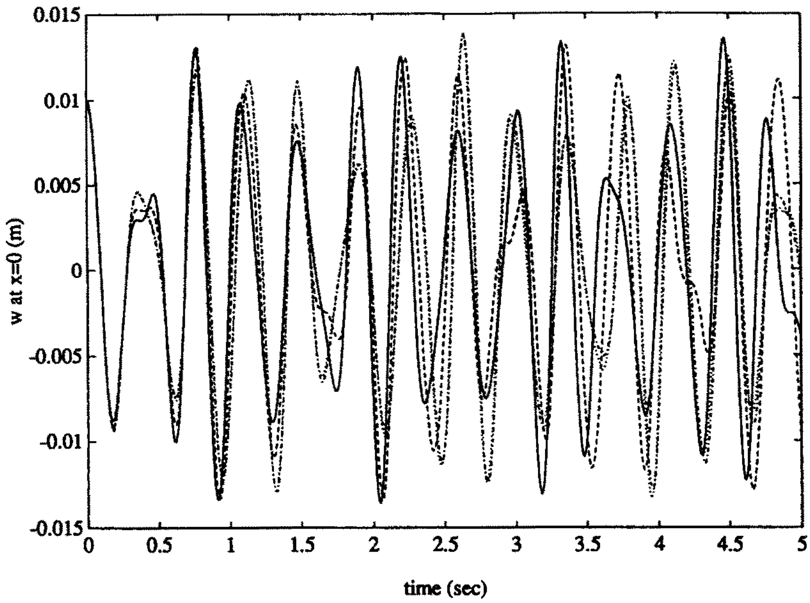

Fig. 4. Tip displacements for a beam moving over supports shown in Fig. 1, $\Omega = 10$ rad s^{-1}, ——, $n = 4$, - - - -, $n = 5$, · · · · · ·, $n = 6$, - · - · - · -, $n = 7$.

different from the behaviors of the beam shown in Figs 4 and 6. As in the case of a beam moving over supports shown in Fig. 1, the stability of the beam moving over supports shown in Fig. 2 is also found to be dependent on the prescribed value of Ω.

Repositioning maneuver

For a prescribed repositioning maneuvering of the beam, the variation of $x = a$ is assumed to be

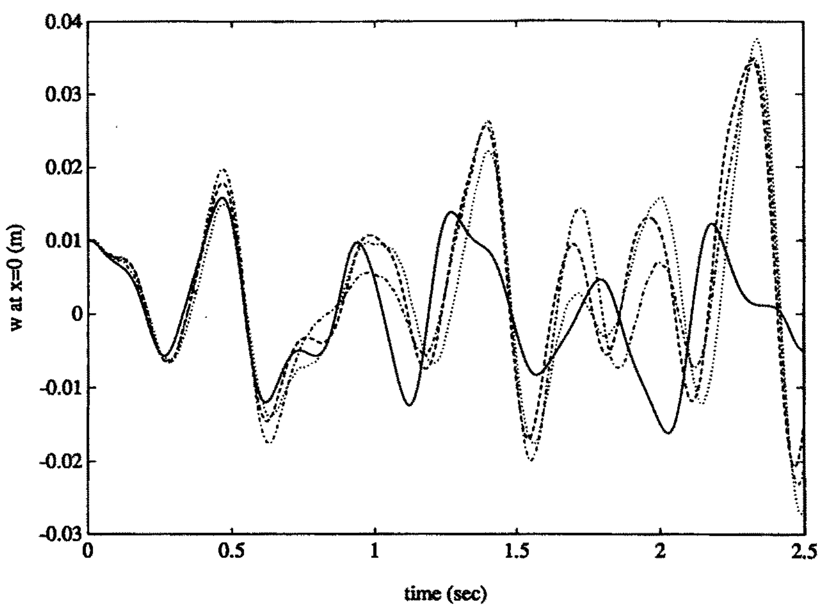

Fig. 5. Tip displacements for a beam moving over supports shown in Fig. 1, $\Omega = 20$ rad s^{-1}, ——, $n = 4$, - - - -, $n = 5$, · · · · · ·, $n = 6$, - · - · - · -, $n = 7$.

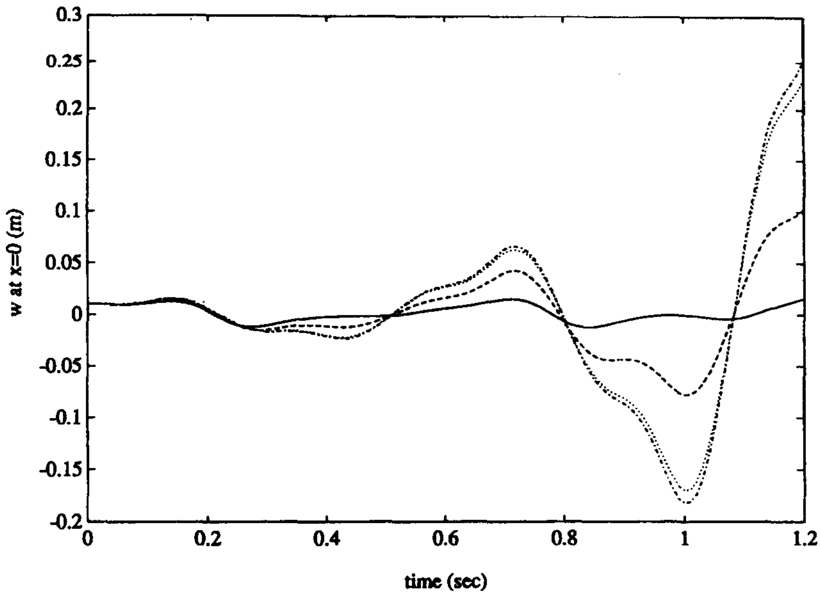

Fig. 6. Tip displacements for a beam moving over supports shown in Fig. 1, $\Omega = 22$ rad s^{-1}, ——, $n = 4$, - - - -, $n = 5$, · · · · · ·, $n = 6$, - · - · - · -, $n = 7$.

$$a = C_1 - \frac{C_2}{T_d}\left[t - \frac{T}{2\pi}\sin\frac{2\pi t}{T_d}\right]\text{m}, \tag{32}$$

where C_1 specifies the initial position of the beam, C_2 is the horizontal distance traversed by any point on the beam due to the assumption of axial rigidity, and T_d is the duration of the prescribed motion. For the same prescribed initial position of the beam as in the longitudinal sinusoidal motion, $C_1 = 0.375$ m.

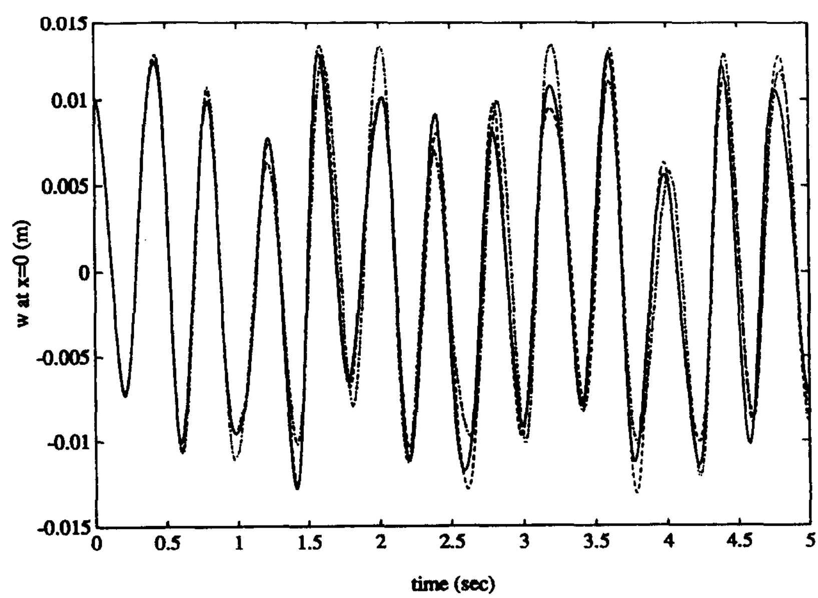

Fig. 7. Tip displacements for a beam moving over supports shown in Fig. 1, $\Omega = 20$ rad s^{-1}, $A = 0.025$, ——, $n = 4$, - - - -, $n = 5$, · · · · · ·, $n = 6$, - · - · - · -, $n = 7$.

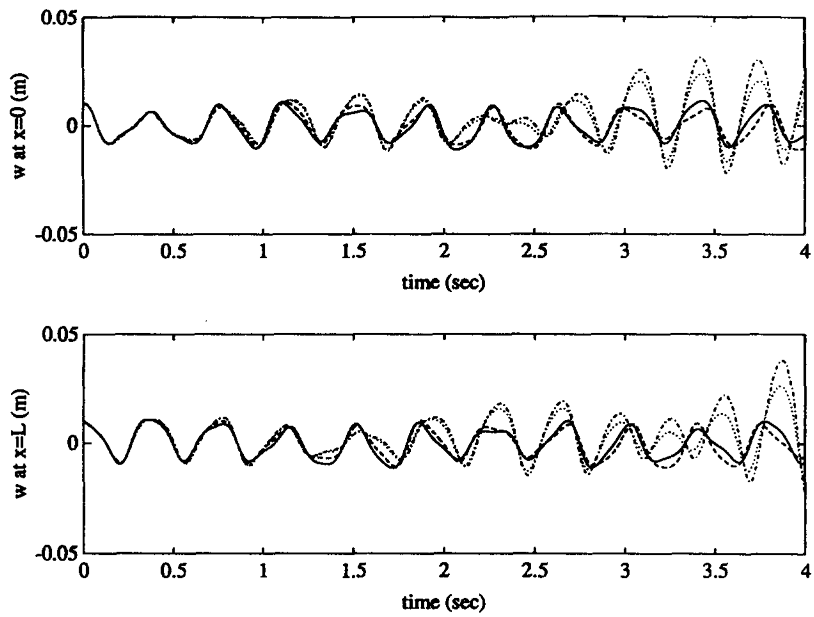

Fig. 8. Tip displacements for a beam moving over supports shown in Fig. 2, $\Omega = 20$ rad s^{-1}, ——, $n = 4$, - - - -, $n = 5$, · · · · · ·, $n = 6$, -·-·-·-·, $n = 7$.

The tip displacements of the beam for repositioning maneuvering with $C_2 = 0.35$ m and $T_d = 0.35$ s are shown in Fig. 11 for both cases of support configurations shown in Figs 1 and 2. It can be seen from Fig. 11 that there are some noticeable differences in the two behaviors of the beam for such a "fast" prescribed motion of the beam. On the other hand, for a relatively "slow" prescribed motion with $C_2 = 0.35$ m and $T_d = 3.5$ s, the two behaviors of the beam, shown in Fig. 12, appear to be almost identical for the two different support configurations. The apparent reduction in the period as well as amplitude of vibration for the tip displacement at $x = 0$ is due to the decreasing distance between the free end at $x = 0$ and the left support at $x = a$.

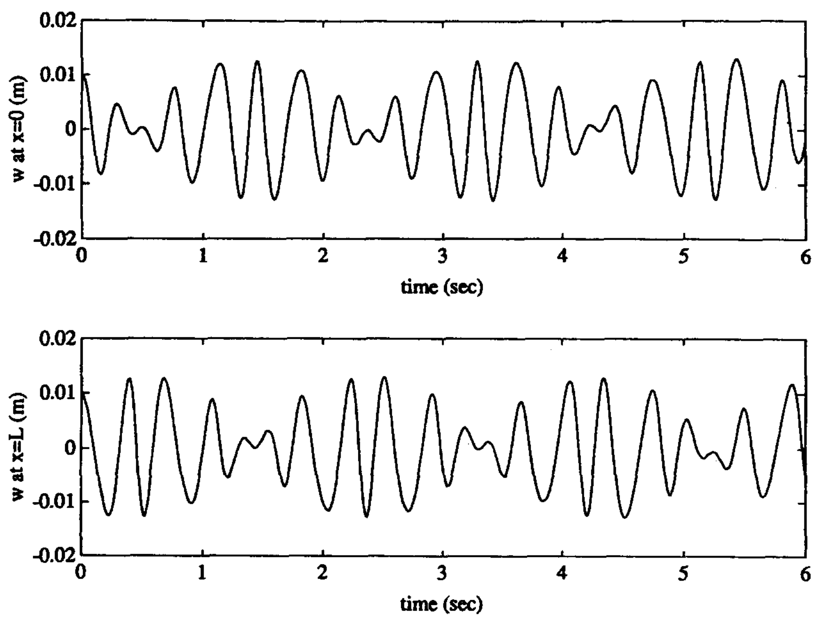

Fig. 9. Tip displacements for a beam moving over supports shown in Fig. 2, $\Omega = 10$ rad s^{-1}, $n = 6$.

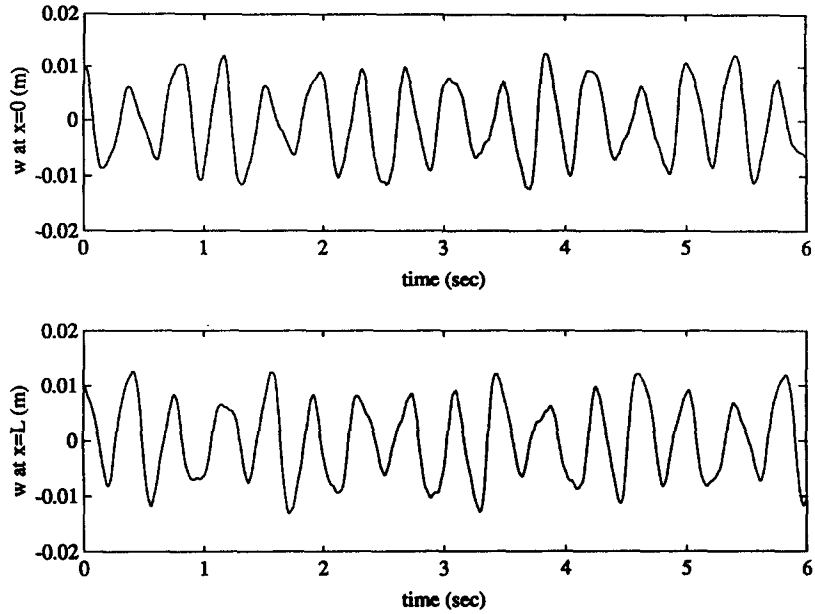

Fig. 10. Tip displacements for a beam moving over supports shown in Fig. 2, $\Omega = 22$ rad s^{-1}, $n = 6$.

4. CONCLUSION

Approximate equations of motion in matrix form are derived for the motion of a beam moving over multiple supports using Hamilton's principle and the assumed mode method. The external forces that cause the motion of the beam are either in the form of the frictional forces supplied by a roller which acts as a support at the same time, or external forces applied at one end of the beam. For longitudinal sinusoidal motion of the beam, numerical simulations show that the behaviors of the beam with these two different modes of applied forces can be dramatically different depending on the frequency of the prescribed longitudinal excitation. For repositioning maneuvering of the beam, the behavior of the beam also changes with different modes of applied external forces for relatively fast prescribed

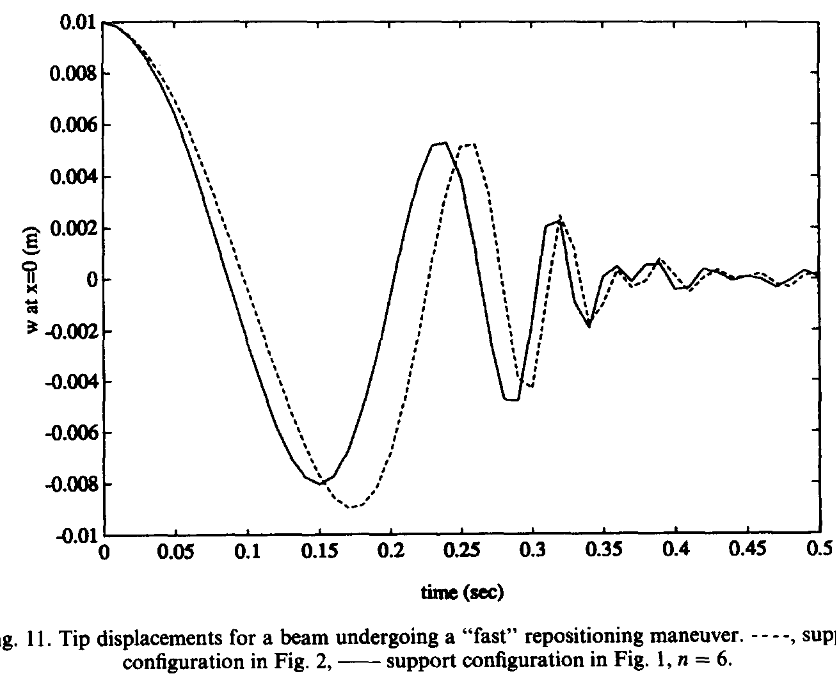

Fig. 11. Tip displacements for a beam undergoing a "fast" repositioning maneuver. ----, support configuration in Fig. 2, —— support configuration in Fig. 1, $n = 6$.

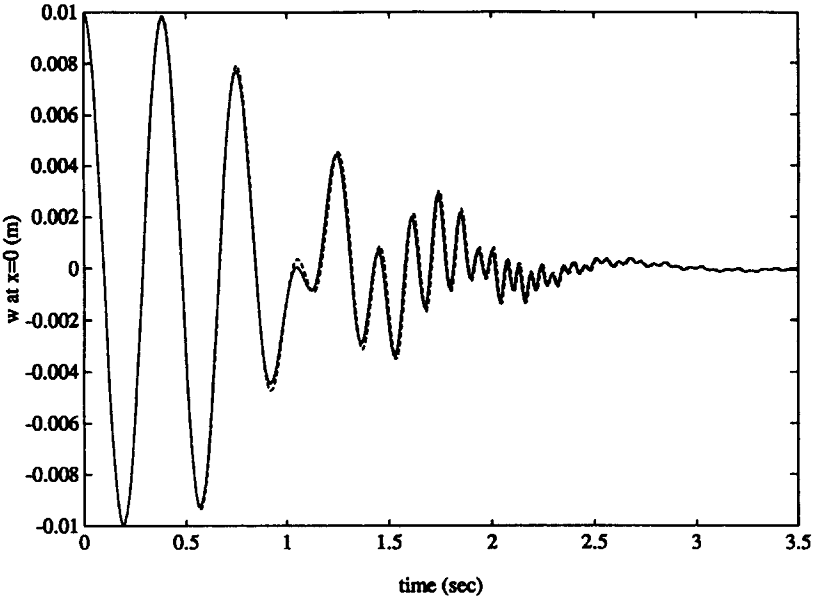

Fig. 12. Tip displacements for a beam undergoing a "slow" respositioning maneuver. - - - -, support configuration in Fig. 2, —— support configuration in Fig. 1, $n = 6$.

motions of the beam. On the other hand, the behaviors of the beam appear to be the same for different modes of external applied forces if the prescribed longitudinal motion of the beam is over a relatively long duration.

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